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Kac-Moody algebras in perturbative string theory

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Abstract: The conjecture that M-theory has the rank eleven Kac-Moody symmetry e_{11} implies that type-IIA and type-IIB string theories in ten dimensions possess certain infinite dimensional perturbative symmetry algebras that we determine. This prediction is compared with the symmetry algebras that can be constructed in perturbative string theory, using the closed string analogues of the DDF operators. Within the limitations of this construction close agreement is found. We also perform the analogous analysis for the case of the closed bosonic string.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, M-Theory, Space-Time Symmetries.

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1. Introduction

During the last century our understanding of particle physics has been transformed by the introduction of group theory. Indeed we are now sure that all the forces of nature, except for gravity, are described by Yang-Mills gauge theories based on finite dimensional Lie groups. At low energies these symmetries are spontaneously broken or confined, and are therefore not obviously visible. A significant step towards the identification of the underlying symmetries was the demonstration that if a global symmetry G is spontaneously broken, then the resulting massless modes are very often described by a non-linear realisation based on G, with a subgroup H that is preserved by the symmetry breaking mechanism. Thus although one might not know the underlying theory and the mechanism of symmetry breaking, one could find the groups involved by comparing the known data with the results of the theory of non-linear realisations.

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Gravity is described, at low energies, by Einstein's theory which, unlike the other forces, is not a Yang-Mills gauge theory. We now believe that a consistent theory of gravity requires supersymmetry and string theory as essential ingredients. Our belief in this is supported by the realisation that, if one adopts this view point, gravity becomes automatically unified with the other forces. The relevant maximal supergravity theories, i.e. the IIA [1, 2, 3] and IIB [4, 5, 6] supergravity theories in ten dimensions, are uniquely determined by supersymmetry. They define the complete low energy effective theory of type-IIA and -IIB superstring theory. In turn, the two string theories are believed to be suitable limits of M-theory which has eleven-dimensional supergravity as its low energy effective action [7]. Given the important role of symmetry in particle physics it would be reasonable to suppose that M-theory should also possess a very large symmetry group.

The symmetry algebras used in particle physics are based on finite dimensional Lie algebras. More recently, the discovery of Kac-Moody and Borcherds algebras [8, 9, 10] has considerably enlarged the class of Lie algebras beyond that previously considered. Although the vertex operator technology of string theory has played a significant part in the mathematical development of these algebras, only a subset of these, affine Kac-Moody algebras, have played an important role in string theory and conformal field theory so far.

Recently it has been shown [11, 12] that the bosonic sectors, including gravity, of all maximal supergravity theories can be described by a non-linear realisation. Unlike the usual treatments of gauge fields and gravity, this procedure treats the different fields on the same footing, namely as Goldstone bosons for a certain symmetry group. The nature of this construction points to the possibility that there exist alternative formulations of these supergravity theories that are invariant under a Kac-Moody (and possibly a Borcherds) algebra. Although this was not proved in reference [12], it was possible to show that, if this were the case, then the symmetry must include the Kac-Moody algebra e_{11} for the case of eleven dimensional supergravity and IIA supergravity. Furthermore, it was shown that the corresponding Kac-Moody algebra for the IIB theory was also e_{11} [13]. This is consistent with the idea that these different string theories are part of a single theory, namely M-theory, that also has e_{11} as a symmetry. The different string theories correspond then to different choices of a 'vacuum' in this theory, together with different subgroups that preserve the vacuum under consideration.

In all cases, the e_{11} symmetry contains the Lorentz group which is evidently a symmetry of perturbation theory. However, it also contains symmetries which are non-perturbative. The simplest example is provided by the case of IIB where the e_{11} symmetry contains the sl(2) symmetry [4] of IIB supergravity. Since this symmetry acts on the dilaton, it mixes perturbative with non-perturbative phenomena. The e_{11} symmetries therefore also contain in general a mixture of perturbative and non-perturbative symmetries.

In the examples of non-linear realisations found in particle physics, only the preferred subgroup is linearly realised in the non-linear realisation. However, in these examples the non-linear realisation only arises as a low energy effective action of some more fundamental theory, and in this fundamental theory the full symmetry group is linearly realised. This gives rise to the expectation that in M-theory, the e_{11} algebra is actually linearly realised. If this is the case, then a suitable subalgebra of this symmetry will also be realised linearly in string theory.

In this paper we want to explore some of the consequences of the conjecture that M-theory is invariant under the e_{11} symmetry. In particular, we want to determine the perturbative symmetries of the IIA and IIB superstring theories that are predicted by this assumption. Since the symmetry breaking of the e_{11} algebra is different for the two theories, we find different perturbative subalgebras in the two cases. We shall also perform the analogous analysis for the closed bosonic string that has been proposed to possess the symmetry k_{27} [12].

If these symmetries are indeed linearly realised in M-theory (or the full closed bosonic string theory), the perturbative subalgebras should also be linearly realised in the appropriate string theories, and should therefore be accessible. In order to analyse whether these symmetries are indeed present, we perform a DDF construction for the different closed string theories. Unlike the open string case where single DDF operators act on the space of physical states, individual DDF operators do not preserve the space of physical states of the closed string. However, one can construct bilinear combinations of DDF operators that map physical states to physical states. These bilinear combinations define symmetries of the string scattering amplitudes, and they generate certain affine Kac-Moody algebras. The idea that DDF operators may be important for the construction of symmetries in string theory was first suggested in [14].

The algebras that are generated by these bilinear combinations of DDF operators can be determined for the different cases, and we find considerable agreement with the above predictions. We regard this as good evidence for the conjecture that M-theory is e_{11} invariant and that the closed bosonic string has a k_{27} symmetry.

The paper is organised as follows. In section 2 we determine the perturbative subalgebras of the full symmetry algebras for the different cases. The DDF construction for the closed bosonic string, and the determination of the corresponding symmetry algebra is performed in section 3. Section 4 deals with the generalisation of this construction to the supersymmetric case. Finally, in section 5 we explain that the full symmetry algebras that arise are of a rather special kind that is associated with even self-dual lattices, and we speculate about the possibility of an 18-dimensional string theory. There is one short appendix that describes our conventions for the simple roots of e_8.

2. Perturbative symmetries

It was argued in [12] that M-theory possesses a large symmetry that contains e_{11}. As is well known, the circle compactification of M-theory is described by type-IIA string theory [15], while type-IIB can be obtained upon compactifying M-theory on a torus and taking the volume of the torus to zero size [16]. From the point of view of type-IIA and type-IIB string theory, the symmetries described by e_{11} combine perturbative as well as non-perturbative symmetries. In this section we want to determine the *perturbative* symmetries of type-IIA and type-IIB that are predicted in this way. As we shall see, a large part of these symmetries

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can be understood from a perturbative string construction that we shall present in later sections. We shall also carry out the analogous calculation for the closed bosonic string which was argued to possess a k_{27} symmetry.

In closed string theory the string coupling constant g_s is related to the expectation value of the dilaton field, ϕ, as $g_s = \exp\phi$. The perturbative symmetries are therefore characterised by the property that they do not transform the dilaton field. In the approach of reference [12] the supergravity theories are constructed as a non-linear realisation. Every field is therefore a Goldstone field and so is associated with a generator in $\mathfrak{g} = e_{11}$ or $\mathfrak{g} = k_{27}$. Let us denote the generator associated with the dilaton by R_I where $I = A$ for the case of type IIA, $I = B$ for type IIB and $I = CB$ for the closed bosonic string. Within this framework, the perturbative symmetry algebra $\mathfrak{p}_I$ is then the subalgebra of $\mathfrak{g}$ that commutes with the generator R_I, modulo transformations proportional to R_I; more specifically

$$\mathfrak{p}_I = \{a \in \mathfrak{g} : [a, R_I] = 0\}/R_I\,. \tag{2.1}$$

The bracket on $\mathfrak{p}_I$ is that inherited from $\mathfrak{g}$. The explicit expression for the generator R_I is different for the three cases, and we shall therefore consider them in turn.

2.1 The case of type IIA

As was shown in [12, eq. (4.21)] (see also [11]), the generator associated with the type-IIA dilaton is given by

$$R_A = \frac{1}{12}\left(-\sum_{a=1}^{10} K^a{}_a + 8K^{11}{}_{11}\right). \tag{2.2}$$

This generator is an element of the Cartan subalgebra of e_{11} and can be written in terms of the Cartan generators as

$$R_A = \frac{1}{4}H_8 - \frac{1}{4}H_6 - \frac{1}{2}H_7\,, \tag{2.3}$$

where we have adopted the numbering of [17] that is illustrated in figure 1. Given (2.3), the commutators of R_A with the positive simple roots of e_{11} are

$$\begin{aligned}
[R_A, E_a] &= 0\,, \qquad a = -2, \ldots, 6\,,\\
[R_A, E_7] &= -\frac{3}{4}\,E_7\,,\\
[R_A, E_8] &= \frac{1}{2}\,E_8\,,
\end{aligned} \tag{2.4}$$

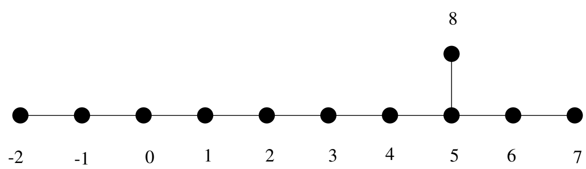

Figure 1: The Dynkin diagram of e_{11}. We have adopted the conventions of [17], describing e_{11} as the very-extended e_8 algebra.

together with the same relations (with opposite signs) for the negative simple roots F_a. The remaining roots of the Kac-Moody algebra e_{11} are given by the multiple commutators

$$[E_{a_1}, [\ldots [E_{a_{p-1}}, E_{a_p}] \ldots] , \tag{2.5}$$

subject to the Serre relations, as well as similar multiple commutators of the F_a. The multiple commutator in (2.5) describes the positive root

$$\sum_{a=-2}^{8} n_a\, \alpha_a , \tag{2.6}$$

where α_a denotes the root corresponding to E_a, and n_a is the non-negative integer that describes the multiplicity with which E_a appears in (2.5). The root corresponding to the multiple commutator of the F_a is also described by (2.6), except that now all n_a are non-positive integers.

The roots of the perturbative subalgebra $\mathfrak{p}_A$ are the subset of the roots in (2.6) for which $(-3/4)n_7 + (1/2)n_8 = 0$, as follows from (2.4). The positive roots of $\mathfrak{p}_A$ therefore consist of the roots of e_{11} that are of the form

$$\sum_{a=-2}^{6} n_a\, \alpha_a + m\,(2\,\alpha_7 + 3\,\alpha_8) , \tag{2.7}$$

where $n_a, m \in \mathbb{Z}_{\geq 0}$, while the negative roots are given by (2.7) with $n_a, m \in \mathbb{Z}_{\leq 0}$.

In order to discover what this algebra actually is, it is useful to express the roots (2.7) in terms of a set of simple roots. If we define

$$\beta_A = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8 , \tag{2.8}$$

then $\beta_A^2 = 2$, $\beta_A \cdot \alpha_0 = -1$, $\beta_A \cdot \alpha_6 = -1$, while $\beta_A \cdot \alpha_a$ with $a = -2, -1, 1, 2, \ldots, 5$ vanish. Thus we can choose the set of simple roots to consist of

$$\alpha_a , \quad a = -2, \ldots, 6 , \qquad \text{and} \qquad \beta_A . \tag{2.9}$$

The corresponding Dynkin diagram is given in figure 2; it is precisely the Dynkin diagram of the very-extended su(8) Kac-Moody algebra [17].[1] The perturbative subalgebra contains

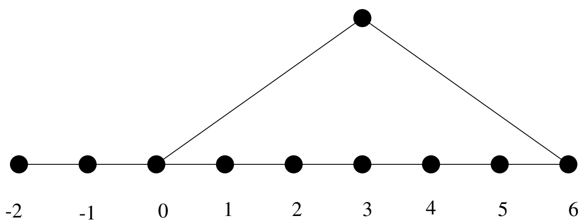

Figure 2: The Dynkin diagram of the perturbative $\mathfrak{p}_A$ algebra that is isomorphic to the very-extended su(8) Kac-Moody algebra.

[1] To every finite dimensional simple Lie algebra $\mathfrak{g}$ of rank r, one can construct a lorentzian Kac-Moody algebra of rank $r + 3$. The resulting Lie algebra is called the very-extended algebra of $\mathfrak{g}$; further details can be found in [17].

the sub-algebra generated by the simple roots α_a, $a=-2,\ldots,6$ which corresponds to the SL(10) subgroup in the non-linear realisation whose Goldstone boson is the graviton.

We now give an alternative derivation of $\mathfrak{p}_A$, using the description of the root lattice of e_{11} given in [17]. For this it is useful to consider the Cartan-Weyl basis of the algebra. Recall that the Cartan generators in the Chevalley and the Cartan-Weyl basis are related as $H_a = \frac{2}{(\alpha_a,\alpha_a)}\alpha_a^i H_i$, and that in the Cartan-Weyl basis the generator associated with the root α obeys $[H_i, E_\alpha] = \alpha_i E_\alpha$. In terms of the Cartan-Weyl basis of the algebra, the generator associated with the dilaton field, given in equation (2.3), can therefore be expressed as $R_A = (\alpha_A)^i H_i$, where

$$\alpha_A = \frac{1}{4}\alpha_8 - \frac{1}{4}\alpha_6 - \frac{1}{2}\alpha_7 \,. \tag{2.10}$$

The roots of $\mathfrak{p}_A$ are then the roots of e_{11} that are orthogonal to α_A.

Next we use the fact that the root lattice, $\Lambda_{e_{11}}$, of e_{11} is the lattice

$$\Lambda_{e_{11}} = \Lambda_{e_8} \oplus \mathrm{II}^{1,1} \oplus \left\{x \in \mathrm{II}^{1,1} : x.s = 0\right\}, \tag{2.11}$$

where s is a time-like vector in the second $\mathrm{II}^{1,1}$. Since $\alpha_A \in \Lambda_{e_8}$, the root lattice of $\mathfrak{p}_A$ is then

$$\Lambda_{\mathfrak{p}_A} = \{x \in \Lambda_{e_8} : x.\alpha_A = 0\} \oplus \mathrm{II}^{1,1} \oplus \left\{x \in \mathrm{II}^{1,1} : x.s = 0\right\}. \tag{2.12}$$

The first lattice of the right hand side is actually the root lattice of the subalgebra of e_8 that commutes with $R_A \in e_8$; in particular, it is therefore clear that $\mathfrak{p}_A$ is a very-extended Kac-Moody algebra [17]. In order to determine what the commutant of R_A in e_8 is, we observe that in the basis of appendix A,

$$2\,\alpha_A = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right). \tag{2.13}$$

It is now easy to see that the roots in Λ_{e_8} that are orthogonal to α_A are of the form

$$\pm\left(1,1,0^6\right), \qquad \pm\left(0,1,-1,0^5\right), \tag{2.14}$$

where in the first case the second 1 can take any of the remaining seven places, while in the second case both 1 and -1 can take any two of the seven remaining places. It is easy to check that these are precisely the roots of su(8). Thus we have shown that $\mathfrak{p}_A$ is the very-extended su(8) algebra, in agreement with the derivation above.

2.2 The case of type IIB

The analysis for the case of type IIB is completely analogous, and we shall therefore be rather brief. The generator corresponding to the type-IIB dilaton was determined in [13], and it is given in terms of the roots of e_{11} as $R_B = H_7$ (in the Chevalley basis) or as $\alpha_B = \alpha_7$ (in the Cartan-Weyl basis). In terms of the basis for e_8 of appendix A, we therefore have

$$\alpha_B = \alpha_7 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). \tag{2.15}$$

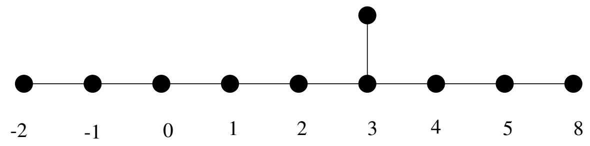

Figure 3: The Dynkin diagram of the perturbative $\mathfrak{p}_B$ algebra that is isomorphic to the very-extended e_7 Kac-Moody algebra.

Using a similar analysis as in the case of IIA one finds that the subalgebra of e_8 that commutes with α_B is actually e_7.[2] Since $\alpha_B \in e_8$, it then follows, as above, that $\mathfrak{p}_B$ is the very-extended Kac-Moody algebra corresponding to e_7. More specifically, we can choose the simple roots of $\mathfrak{p}_B$ to be given by

$$\alpha_a\,, \quad a = -2, \ldots, 5, 8\,, \qquad \text{and} \qquad \beta_B\,, \tag{2.16}$$

where

$$\beta_B = \alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8\,. \tag{2.17}$$

It is easy to see that $\beta_B^2 = 2$, $\beta_B \cdot \alpha_3 = -1$, while $\beta_B.\alpha_a = 0$ for $a = -2, -1, \ldots, 2, 4, 5, 8$. The corresponding Dynkin diagram is shown in figure 3.

2.3 Some comments

It is also instructive to determine the symmetry algebra that is simultaneously perturbative with respect to both type IIA and type IIB. The subalgebra of e_8 that commutes with both α_A and α_B is actually equal to su(7), and thus it follows that the common perturbative subalgebra is the very-extended algebra of su(7). This Lie algebra contains su(9), which has a natural interpretation in terms of the compactification of M-theory on $T^2 \times T^9$. If we are interested in symmetries that preserve the dilaton of IIA and IIB, this implies that the moduli of the T^2 are frozen. The remaining perturbative symmetries are then those of a 9-torus, thus giving rise to $SL(9, \mathbb{Z})$; the corresponding continuous group is thus SU(9).

It is amusing to observe that e_{11} sits rather naturally inside $\mathrm{II}^{10,2}$ (see (2.11) above). The lattice $\mathrm{II}^{10,2}$ is obviously reminiscent of F-theory [19]. From this point of view the direction corresponding to the vector s in (2.11) plays the role of the time-like direction on which one has to compactify F-theory in order to obtain M-theory. It is also somewhat suggestive that the vectors (2.13) and (2.15), whose orthogonal complement define the perturbative subalgebras for IIA and IIB, respectively, are 'spinor weights' of opposite chirality. It would be very interesting to understand these observations more conceptually. It would also be interesting to understand the relation of the e_{11} symmetry of M-theory to those obtained recently in [20, 21].

2.4 The closed bosonic string

The low energy effective action for the closed bosonic string is thought to be invariant under the Kac-Moody algebra k_{27} of rank 27 [12]. Its Dynkin diagram is shown in figure 4.

[2] This also follows from the statement on page 124 of [18].

Figure 4: The Dynkin diagram of the algebra k_{27} that is isomorphic to the very-extended d_{24} Kac-Moody algebra.

The dilaton is the Goldstone boson which is associated to the generator R_{CB} that is an element of the Cartan subalgebra. Its commutation relations with the Chevalley generators of k_{27} are given as [12]

$$\begin{aligned} [R_{CB}, E_a] &= 0\,, \qquad a = -2,-1,0,2,3,\ldots,23\,, \\ [R_{CB}, E_1] &= -E_1\,, \\ [R_{CB}, E_{24}] &= E_{24}\,, \end{aligned} \tag{2.18}$$

together with the same relations (with the opposite signs) with E_a being replaced by F_a. Using the same arguments as above for the case of type IIA, the positive roots of $\mathfrak{p}_{CB}$ then consist of multiple commutators of the E_a that contain E_1 and E_{24} with the same multiplicity. If we define

$$\beta_{CB} = \sum_{a=2}^{22} \alpha_a + \alpha_1 + \alpha_{24}\,, \tag{2.19}$$

then $\beta^2 = 2$, $\beta.\alpha_0 = -1$, $\beta.\alpha_{23} = -1$ and $\beta.\alpha_a = 0$ for $a = -2,-1,2,3,\ldots,22$. As such,

$$\alpha_a\,, \quad a = -2,-1,0,2,3,\ldots,23\,, \qquad \text{and} \qquad \beta \tag{2.20}$$

form a set of simple roots for the perturbative algebra. It is easy to see that the corresponding Dynkin diagram is that of the very-extended a_{23} algebra. It contains the subalgebra generated by the simple roots α_a, $a = -2,-1,0,2,3,\ldots,23$ which corresponds to the SL(26) sub in the non-linear realisation whose Goldstone boson is the graviton.

3. Kac-Moody symmetries in the bosonic string

We now want to show that the space of states of the closed string theory naturally carries an action of a certain Kac-Moody algebra that we shall construct. We shall also explain that this is a symmetry of the perturbative string scattering amplitudes. The construction makes use of the DDF-operators [22]. We shall first consider the closed bosonic string; the analogous analysis for the superstring will be described in the next section.

3.1 The DDF construction of the open bosonic string

The DDF operators are usually discussed within the context of the open (bosonic) string; it will therefore be instructive to review this construction briefly before generalising it to the closed bosonic string. The open bosonic string sweeps out a surface in space-time described by the fields $X^\mu(\tau, \sigma)$, where τ and σ are the time- and space-parameters of the world-sheet. It is convenient to rewrite the dynamical variable $X^\mu(\tau,\sigma)$ in terms of the variables $q^\mu \equiv q^\mu(\tau=0)$ and the oscillator modes $\alpha^\mu_n \equiv \alpha^\mu_n(\tau=0)$ as

$$X^\mu(\tau,\sigma) = q^\mu + \sqrt{2\alpha'}\alpha^\mu_0\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n e^{-in(\tau+\sigma)} + \alpha^\mu_n e^{-in(\tau-\sigma)}\right). \tag{3.1}$$

The space-time momentum p^μ is given by $\alpha^\mu_0 = \sqrt{2\alpha'}p^\mu$ and one usually chooses $2\alpha' = 1$. The field $Q^\mu(\tau) = X^\mu(\tau,0)$ can then be written as

$$Q^\mu(z) = q^\mu - ip^\mu \ln z + i\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, z^{-n}\,, \tag{3.2}$$

where $z = e^{i\tau}$.

In terms of the usual light-cone coordinates $p^\pm = \frac{1}{\sqrt{2}}(p^{25} \pm p^0)$ we then choose a particular Lorentz frame by taking the tachyonic ground state to have the momentum $p^+_0 = p^-_0 = 1$, $p^i_0 = 0$. The transverse DDF operators [22] are then defined by

$$A^i_n = \frac{1}{2\pi i}\oint_0 i\partial Q^i(z)\, e^{ink_0\cdot Q(z)}\, dz\,, \tag{3.3}$$

where $k^-_0 = 1$, and $k^+_0 = k^i_0 = 0$. In writing (3.3) we do not have to worry about a normal ordering prescription since $k_0^2 = 0$, and the spatial components of k^μ_0 vanish. The contour integral is well-defined provided that A^i_n acts on states whose momentum has integer inner product with k_0. In particular, this condition is satisfied for the ground tachyon ground state with momentum p_0, and any state that is created from it by the action of the DDF operators (as the DDF operator A^i_n changes the momentum by nk_0).

Using standard techniques it is easy to show that the DDF operators obey the relations of a u(1) current algebra

$$[A^i_m, A^j_n] = m\ \delta^{ij}\ \delta_{m,-n}\,. \tag{3.4}$$

The DDF operators commute with the Virasoro generators $L_n = \frac{1}{2}\sum_m :\alpha_m\cdot\alpha_{n-m}:$ and, as a result, we can create physical states by acting with polynomials of the DDF operators on the tachyonic ground state $|p_0\rangle$. All of these states are annihilated by L_n with $n > 0$ and satisfy the mass-shell condition $L_0\psi = \psi$. In the critical dimension 26, all physical states with momentum $p_0 + Nk_0$ can be created in this way.

3.2 The DDF construction of the closed bosonic string

Let us now generalise the DDF construction to the 26-dimensional closed bosonic string theory. The fields X^μ that describe the embedding of the string in the space-time can now be written as

$$X^\mu(\tau,\sigma) = q^\mu + \sqrt{2\alpha'}\alpha^\mu_0\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n e^{-in(\tau+\sigma)} + \bar{\alpha}^\mu_n e^{-in(\tau-\sigma)}\right), \tag{3.5}$$

where the space-time momentum p^μ is given by $\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}\,p^\mu$. It is traditional to take $\alpha' = 2$. The physical states Ψ of the theory satisfy the mass-shell condition,

$$L_0\Psi = \bar{L}_0\Psi = \Psi\,, \tag{3.6}$$

as well as the Virasoro constraints

$$L_n \Psi = \bar{L}_n \Psi = 0 \qquad \text{for all } n > 0\,. \tag{3.7}$$

Here L_n and $\bar{L}_n$ are defined as $L_n = \frac{1}{2}\sum_m : \alpha_m \cdot \alpha_{n-m} :$ and $\bar{L}_n = \frac{1}{2}\sum_m : \bar{\alpha}_m \cdot \bar{\alpha}_{n-m} :$. In defining $\bar{L}_n$ we have used the mode $\bar{\alpha}_0^\mu$, which at this stage simply equals $\alpha_0^\mu = \bar{\alpha}_0^\mu$. The oscillator modes satisfy the commutation relations

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= m\, \delta^{\mu\nu}\, \delta_{m,-n}\,, \\
[\alpha_m^\mu, \bar{\alpha}_n^\nu] &= 0\,, \\
[\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] &= m\, \delta^{\mu\nu}\, \delta_{m,-n}\,.
\end{aligned} \tag{3.8}$$

In order to describe the space of states of the closed string it is convenient to introduce separate left- and right-moving momentum operators $p_L = \alpha_0$ and $p_R = \bar{\alpha}_0$ without imposing initially that they agree, as well as the corresponding position operators q_L, q_R.[3] These operators satisfy the commutation relations

$$\begin{aligned}
[q_L^\mu, p_L^\nu] &= i\eta^{\mu\nu}\,, \\
[q_L^\mu, q_R^\nu] &= [q_L^\mu, p_R^\nu] = [q_R^\mu, p_L^\nu] = [p_L^\mu, p_R^\nu] = 0\,, \\
[q_R^\mu, p_R^\nu] &= i\eta^{\mu\nu}\,,
\end{aligned} \tag{3.9}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. As before for the case of the open string we introduce the fields

$$\begin{aligned}
X_L^\mu(z) &= q_L^\mu - i p_L^\mu \ln z + i \sum_{n\neq 0} \frac{1}{n}\, \alpha_n^\mu\, z^{-n}\,, \\
X_R^\mu(z) &= q_R^\mu - i p_R^\mu \ln \bar{z} + i \sum_{n\neq 0} \frac{1}{n}\, \bar{\alpha}_n^\mu\, \bar{z}^{-n}\,.
\end{aligned} \tag{3.10}$$

The field of equation (3.5) can then be expressed as

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z})\,, \tag{3.11}$$

where $z = \tau + \sigma$ and $\bar{z} = \tau - \sigma$ provided we take $p_L^\mu = p_R^\mu = p^\mu$ and $q^\mu = q_L^\mu + q_R^\mu$. The actual space of states is therefore the subspace of our larger space where we impose the 'reality condition' $p_L^\mu = p_R^\mu$. This larger space is generated by the action of the modes α_n^μ and $\bar{\alpha}_n^\mu$ with $n < 0$ on the ground states $|(p_L, p_R)\rangle$, where (p_L, p_R) denotes the ground state momentum, i.e.

$$p_L^\mu\, |(k_L, k_R)\rangle = k_L^\mu\, |(k_L, k_R)\rangle\,, \qquad p_R^\mu\, |(k_L, k_R)\rangle = k_R^\mu\, |(k_L, k_R)\rangle\,. \tag{3.12}$$

The ground states are annihilated by the oscillators α_n^μ and $\bar{\alpha}_n^\mu$ with $n > 0$.

Next we introduce the (chiral) vertex operators

$$\begin{aligned}
V^i(nk_0, z) &= i\partial X_L^i(z)\, e^{ink_0\cdot X_L(z)}\,, \\
\bar{V}^i(nk_0, \bar{z}) &= i\bar{\partial} X_R^i(\bar{z})\, e^{ink_0\cdot X_R(\bar{z})}\,,
\end{aligned} \tag{3.13}$$

[3]From a more abstract point of view this means that we are complexifying the space-time.

where $k_0^- = 1, k_0^+ = 0 = k_0^i$, and as before no normal ordering prescription is required. We then define their 'zero' mode by

$$
\begin{aligned}
A_n^i &= \frac{1}{2\pi i} \oint_0 dz\, V^i(nk_0, z)\,, \\
\bar{A}_n^i &= \frac{1}{2\pi i} \oint_0 d\bar{z}\, \bar{V}^i(nk_0, \bar{z})\,.
\end{aligned}
\tag{3.14}
$$

These operators are the closed string analogues of the open string DDF-operators of equation (3.3). They are well-defined on states whose left- and right-moving momentum has integer inner product with k_0.

We observe that the operators X_L and X_R both have the same form as Q^μ of equation (3.2), and that the operators they contain have the same commutation relations as those in Q^μ. As such, they satisfy simple algebra relations among themselves

$$
\begin{aligned}
[A_m^i, A_n^j] &= m\, \delta^{ij}\, \delta_{m,-n}\,, \\
[A_m^i, \bar{A}_n^j] &= 0\,, \\
[\bar{A}_m^i, \bar{A}_n^j] &= m\, \delta^{ij}\, \delta_{m,-n}\,,
\end{aligned}
\tag{3.15}
$$

by analogy with equation (3.4). Furthermore, for the same reason, they commute with the two Virasoro algebras

$$
\begin{aligned}
[L_m, A_n^i] &= 0\,, \qquad \text{for all } m, n \in \mathbb{Z}\,, \\
[\bar{L}_m, A_n^i] &= 0\,, \qquad \text{for all } m, n \in \mathbb{Z}\,,
\end{aligned}
\tag{3.16}
$$

and similarly for $\bar{A}_n^i$. Here we have defined $\bar{L}_n$ as in the paragraph following (3.7), but we do not now assume that $\bar{\alpha}_0 = \alpha_0$. The DDF operators map states satisfying equation (3.6) and (3.7) into themselves. However, A_n^i changes the left-moving momentum p_L to $p_L + nk_0$, while $\bar{A}_m^i$ changes the right-moving momentum p_R to $p_R + mk_0$. In particular, a single DDF operator with $n \neq 0$ therefore does not preserve the condition $p_L = p_R$. This will play an important role in the following.

The tachyonic ground state is described by $|(p,p)\rangle$ and it satisfies all the physical state conditions (and in particular the mass-shell condition) provided that $p^2 = 2$. As before we choose a fixed vector p_0 with $p_0^2 = 2$ by taking $p_0^+ = p_0^- = 1$ with $p^i = 0$. With this choice $p_0 \cdot k_0 = 1$. The DDF operators with positive moding annihilate the tachyon ground state

$$
A_n^i |(p_0, p_0)\rangle = \bar{A}_n^i |(p_0, p_0)\rangle = 0 \qquad \text{for } n > 0\,,
\tag{3.17}
$$

since the momentum of these states is $(p_0 + nk_0, p_0)$ and $(p_0, p_0 + nk_0)$, respectively, and no state with (left- or right-moving) momentum of the form $p_0 + nk_0$ with $n > 0$ can satisfy equation (3.6) as $(p_0 + nk_0)^2 = 2 + 2n$.

On the other hand, the DDF operators corresponding to negative modes do not annihilate the tachyonic ground state, and lead to the states

$$
\prod_{k=1}^{r} A_{-n_k}^{i_k} \prod_{l=1}^{s} \bar{A}_{-m_l}^{j_l} |(p_0, p_0)\rangle\,.
\tag{3.18}
$$

These states have momentum of the form $(p_0 - n_L k_0, p_0 - n_R k_0)$, where $n_L = \sum_{k=1}^r n_k$ and $n_R = \sum_{l=1}^s m_l$. In particular, the action of the closed string DDF operators is therefore well-defined on these states. Although the resulting states satisfy the conditions of equations (3.6) and (3.7), they will not in general be physical states of the closed bosonic string as they do not necessarily obey $p_L = p_R$. However, they will be physical states if they satisfy

$$n_L = n = n_R\,, \quad \text{where} \quad n_L = \sum_{k=1}^{r} n_k\,, \quad \text{and} \quad n_R = \sum_{l=1}^{s} m_l\,. \tag{3.19}$$

If this is the case, the corresponding state has left- and right-moving momentum given by $p = p_0 - n\,k_0$ where $p^2 = 2(1-n)$. Furthermore, since we have $\alpha_0^\mu = \bar{\alpha}_0^\mu$ on the state, the conditions of equations (3.6) and (3.7) actually imply the physical state conditions.[4]

It follows from the commutation relations (3.15) that the corresponding states have positive norm. It is also easy to see that in the critical dimension these states account for all the physical closed string states with momentum $p = p_0 - n\,k_0$. All physical states (except those with $p = 0$) can be related by a Lorentz transformation to a state of the form of equation (3.18) with $n_L = n_R$.

3.3 Symmetries of the spectrum

In the previous subsection we have given a simple description of the space of states of the closed bosonic string. Now we want to construct operators that map this space into itself. As we have mentioned above, the only single DDF-operators that preserve the level matching condition of equation (3.19) are A_0^i and $\bar{A}_0^i$ that act rather trivially (they are actually equal to p_L^i and p_R^i). Unlike the situation for the open bosonic string, we therefore have to consider products of DDF-operators in order to construct non-trivial operators that respect (3.19). The simplest non-trivial example arises when these operators are bilinear in A_n^i and $\bar{A}_n^i$. Then there are three different generators that preserve the level matching condition

$$L^{ij}(n) = \frac{1}{n} A_{-n}^i A_n^j + \frac{1}{2}\delta^{ij}\,, \qquad \bar{L}^{ij}(n) = \frac{1}{n} \bar{A}_{-n}^i \bar{A}_n^j + \frac{1}{2}\delta^{ij}\,, \tag{3.20}$$

where $n > 0$ (the expressions for n and $-n$ differ only by a constant) as well as

$$K^{ij}(m) = \frac{1}{m} A_{-m}^i \bar{A}_{-m}^j\,, \tag{3.21}$$

where $m \neq 0$. The prefactors of $1/n$ and $1/m$, as well as the terms proportional to δ^{ij} have been introduced for later convenience.

Given the commutation relations of the DDF-operators of equation (3.15), we can determine the commutation relations of the operators $L^{ij}(n)$, $\bar{L}^{ij}(n)$ and $K^{ij}(n)$. Let us

[4] Initially we only know that these states satisfy (3.6) and (3.7) where $\bar{L}_n$ is defined as in the paragraph following (3.7), but without assuming that $\alpha_0^\mu = \bar{\alpha}_0^\mu$. If we have $\alpha_0^\mu = \bar{\alpha}_0^\mu$ on the state, this reproduces then precisely the actual physical state condition.

first consider the commutator of two $L^{ij}(n)$ operators. It is given as

$$\begin{aligned}[L^{ij}(n), L^{kl}(m)] &= \frac{1}{nm}[A^i_{-n}A^j_n, A^k_{-m}A^l_m] \\ &= \frac{1}{nm}\left(A^i_{-n}[A^j_n, A^k_{-m}]A^l_m + A^k_{-m}[A^i_{-n}, A^l_m]A^j_n\right) \\ &= \delta_{n,m}\left(\delta^{jk}L^{il}(n) - \delta^{il}L^{kj}(n)\right), \end{aligned} \tag{3.22}$$

where we have used that $n, m > 0$, and therefore that only two of the four possible commutator terms can be non-trivial. We recognise these commutation relations as those of the Lie algebra u(24). Thus, for fixed n, the modes $L^{ij}(n)$ define the Lie algebra of u(24), while the modes for different n commute. Similarly we find that

$$[\bar{L}^{ij}(n), \bar{L}^{kl}(m)] = \delta_{n,m}\left(\delta^{jk}\bar{L}^{il}(n) - \delta^{il}\bar{L}^{kj}(n)\right). \tag{3.23}$$

and therefore also the modes $\bar{L}^{ij}(n)$ form the Lie algebra of u(24). Furthermore, the modes $L^{ij}(n)$ and $\bar{L}^{ij}(m)$ commute.

Finally, we find that the commutation relations with the $K^{kl}(m)$ generators are given by

$$\begin{aligned}[L^{ij}(n), K^{kl}(m)] &= \delta_{n,m}\delta^{jk}K^{il}(m) - \delta_{n,-m}\delta^{ik}K^{jl}(m), \\ [\bar{L}^{ij}(n), K^{kl}(m)] &= \delta_{n,m}\delta^{jl}K^{ki}(m) - \delta_{n,-m}\delta^{il}K^{kj}(m), \\ [K^{ij}(n), K^{kl}(m)] &= \delta_{n,-m}\left(\delta^{ik}\bar{L}^{jl}(n) + \delta^{jl}L^{ik}(n)\right). \end{aligned} \tag{3.24}$$

In writing these equations we have assumed, without loss of generality, that $n > 0$ while m was unrestricted.

Incidentally, the operators $L^{ij}(n)$, $\bar{L}^{ij}(n)$ and $K^{ij}(m)$ can also be defined in the original space of states, without the need of relaxing $p_L = p_R$ (or complexifying space-time). For example we can define

$$K^{ij}(m) = \frac{1}{m}\frac{1}{2\pi i}\oint dz\,\frac{1}{2\pi i}\oint d\bar{z}\,P^i(z)\,\bar{P}^j(\bar{z}) : \exp(imk_0\cdot X(z,\bar{z})) :, \tag{3.25}$$

where $P^i(z) = i\frac{\partial}{\partial z}X^i(z,\bar{z})$, $\bar{P}^j(z) = i\frac{\partial}{\partial \bar{z}}X^j(z,\bar{z})$, and the coordinates z and $\bar{z}$ are regarded as independent (i.e. not as complex conjugates of one another). One can show that $K^{ij}(m)$, as expressed in equation (3.25), commutes with L_n and $\bar{L}_n$. Furthermore, this definition of $K^{ij}(m)$ agrees, on the actual physical subspace, with that given above in (3.21). Similar formulae also exist for $L^{ij}(n)$ and $\bar{L}^{ij}(n)$, and the commutation relations (3.22)–(3.24) can be derived in this way.

It follows from these commutation relations that, for each fixed $n > 0$, the modes

$$L^{ij}(n), \qquad \bar{L}^{ij}(n), \qquad K^{ij}(n), \qquad K^{ij}(-n) \tag{3.26}$$

form a closed subalgebra, and that the algebras corresponding to different positive values of n commute. For each $n > 0$, there are $4(24)^2 = (48)^2$ modes in equation (3.26), and in fact the algebra generated by these modes is precisely u(48). In order to see this we observe

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that the adjoint representation of u(48) decomposes with respect to u(24) $\oplus$ u(24) into the adjoint representation of u(24) $\oplus$ u(24), as well as two bi-fundamental representations. As we have explained above, the modes $L^{ij}(n)$ and $\bar{L}^{ij}(n)$ generate the adjoint representation of u(24) $\oplus$ u(24), and it follows from (3.24) that both the modes $K^{ij}(n)$ and $K^{ij}(-n)$ transform in the bi-fundamental representation of u(24) $\oplus$ u(24).

Next we want to show that these infinitely many commuting finite Lie algebras actually organise themselves into an affine Kac-Moody algebra. The construction we now present works for the various subalgebras, and in particular for u(48) itself, but we give it for the case of the diagonal u(24) subalgebra of u(24) $\oplus$ u(24). The generators $L^{ij}(n)$ and $\bar{L}^{ij}(n)$ do not change the momentum of a state on which they act (and therefore, in particular, do not change their mass), whereas the generators $K^{ij}(n)$ typically do. If we are interested in perturbative symmetries of the theory, we should therefore only consider the generators $L^{ij}(n)$ and $\bar{L}^{ij}(n)$. Furthermore, it is natural to consider the left-right symmetric combination of these generators (that defines the diagonal subalgebra) since this will have a simple geometrical interpretation. The modes we consider are therefore

$$R^{ij}(n) = L^{ij}(n) + \bar{L}^{ij}(n)\,, \tag{3.27}$$

which satisfy the commutation relations

$$[R^{ij}(n), R^{kl}(m)] = \delta_{n,m}\left(\delta^{jk}R^{il}(n) - \delta^{il}R^{kj}(n)\right), \tag{3.28}$$

as follows from equations (3.24). If we are only interested in these generators, it is not necessary to introduce the δ^{ij} correction term in the definition of L^{ij} and $\bar{L}^{ij}$ in (3.20) since it drops out of (3.22) and (3.23).

For each fixed n, the modes $R^{ij}(n)$ only 'see' the DDF-operators with mode number $\pm n$; it is therefore natural to consider the linear combination

$$R_0^{ij} = \sum_{r=1}^{\infty} R^{ij}(r)\,. \tag{3.29}$$

This operator then acts completely geometrically on the space-time indices of the fields and indeed the part which is anti-symmetric in the (i,j) indices is just the generator of the transverse Lorentz transformations. We should stress that R_0^{ij} is well-defined on the (Fock) space of DDF-states since for any given state of finite momentum, the sum in equation (3.29) terminates. This follows directly from the commutation relations of the DDF-operators, as well as the property that the positive DDF-operators annihilate the ground state.

We can generalise the generator of equations (3.29) as follows. Let ζ be an arbitrary (complex) number with $\zeta \neq 0, 1$. Then we define for any $m \in \mathbb{Z}$

$$R_m^{ij} = \sum_{r=1}^{\infty} \zeta^{rm} R^{ij}(n)\,. \tag{3.30}$$

By the same arguments as above, each of the operators R_m^{ij} is well-defined on the Fock space of DDF-states.

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The modes R_n^{ij} satisfy the commutation relations

$$\begin{aligned}
[R_m^{ij}, R_n^{kl}] &= \sum_{r,s=1}^{\infty} \zeta^{rm+sn} [R^{ij}(r), R^{kl}(s)] \\
&= \sum_{r,s=1}^{\infty} \zeta^{rm+sn} \delta_{r,s} \left[\delta^{jk} R^{il}(r) - \delta^{il} R^{kl}(r)\right] \\
&= \sum_{r=1}^{\infty} \zeta^{r(m+n)} \left[\delta^{jk} R^{il}(r) - \delta^{il} R^{kl}(r)\right] \\
&= \delta^{jk} R_{m+n}^{il} - \delta^{il} R_{m+n}^{kl} \, .
\end{aligned} \tag{3.31}$$

These are precisely the commutation relations of the affine Kac-Moody algebra $\hat{\mathrm{u}}(24)$ with a vanishing central term. The algebra $\hat{\mathrm{u}}(24)$ is not simple since it can be decomposed as

$$\hat{\mathrm{u}}(24) = \hat{\mathrm{su}}(24) \oplus \hat{\mathrm{u}}(1) \, . \tag{3.32}$$

In particular, it therefore contains the simple Kac-Moody algebra $\hat{\mathrm{su}}(24)$. It is worth pointing out that the bilinear combinations of DDF operators that appear in R_n^{ij} were required by the level matching condition (3.19), and that the resulting operators define non-abelian commutation relations; this is in marked contrast to the open string DDF operators that only satisfy the relations (3.4). These non-abelian symmetries are thus characteristic of closed string theories.

The reader may find the occurrence of an arbitrary parameter in the construction of equation (3.30) rather artificial. However, we note that taking $\zeta \to \zeta^q$ is equivalent to replacing R_n^{ij} by R_{qn}^{ij}. This leads to an affine algebra that is isomorphic to the original affine algebra. This ambiguity in defining the generators is always present for an affine algebra without centre, and this is responsible for the arbitrariness in choosing ζ.

3.4 Symmetries of amplitudes

As we have seen in the previous subsection, the space of states of the closed bosonic string carries an action of the Kac-Moody algebra $\hat{u}(24)$. We now want to show that this Kac-Moody algebra acts as a symmetry on the scattering amplitudes of the closed bosonic string. It was argued in reference [14] that the DDF operators could be regarded as symmetry operators in the open bosonic string. Here we generalise this consideration to the closed bosonic string, but unlike [14], which used the 'group theoretic' approach to string theory reviewed in [23], we will use more conventional conformal field theory techniques.

First we recall from the definition (3.14) that each (single) DDF-operator is the zero-mode of a chiral vertex operator of conformal weight one. For amplitudes on the Riemann sphere it therefore follows from usual conformal field theory arguments that

$$0 = \sum_{i=1}^{n} \langle V(\psi_1, z_1) \cdots V(\psi_{i-1}, z_{i-1}) V(A\psi_i, z_i) V(\psi_{i+1}, z_{i+1}) \cdots V(\psi_n, z_n) \rangle \, , \tag{3.33}$$

where A is such a zero mode. In writing (3.33) we have assumed that the action of A is well-defined on each of the n external states. This is simply the statement that their momentum along the light-cone directions lies in the lattice spanned by p_0 and k_0. Thus, in effect, we are assuming now that the string theory has been compactified on the lattice $\mathrm{II}^{1,1}$ in these two directions.

The generators of the $\hat{u}(24)$ symmetry are bilinears in the DDF operators. Applying the above argument to each of the two DDF-operators separately we therefore find that

$$\begin{aligned} 0 = & \sum_{i \neq j} \langle V(\psi_1, z_1) \cdots V(A\psi_i, z_i) \cdots V(B\psi_j, z_j) \cdots V(\psi_n, z_n) \rangle + \\ & + \sum_{i=1}^{n} \langle V(\psi_1, z_1) \cdots V(AB\psi_i, z_i) \cdots V(\psi_n, z_n) \rangle \,, \end{aligned} \tag{3.34}$$

where we have written the two single DDF-operators schematically as A and B. Thus if we define the action of AB on the n-fold tensor product of external states as in (3.34), then the amplitudes are invariant under this action. In this sense the Kac-Moody algebra $\hat{u}(24)$ defines a symmetry of the scattering amplitudes. It may also be worthwhile pointing out that the first sum of (3.34) describes amplitudes where two of the external states are not on-shell (since the single DDF operators A and B do not map physical states to physical states). It is therefore conceivable that these contributions actually decouple.

3.5 Comparison with perturbative symmetry

As we have seen above, the closed bosonic string possesses a perturbative symmetry that contains the affine Kac-Moody algebra $\hat{\mathrm{su}}(24)$. It is interesting to see how this algebra compares with the conjecture that the non-perturbative bosonic string has a k_{27} symmetry. As we explained in section 2, the perturbative subalgebra of k_{27} is the very-extended algebra of su(24).

Our construction of the symmetry algebra $\hat{\mathrm{su}}(24)$ was based on the DDF-construction that is, by its nature, not Lorentz invariant, but preserves only the O(24) subgroup of the Lorentz group that acts on the transverse coordinates. As such one should only expect to see the part of the perturbative subalgebra of k_{27} that is preserved by the O(24) subgroup. In the Dynkin diagram of figure 4, the 25 nodes along the horizontal line of k_{27} correspond to the Lorentz symmetries of $\mathrm{SO},(1,25)$, and the space-time directions used in the light-cone formalism are associated with the two nodes that are on the far left. Thus we should only expect to see the truncation of the perturbative subalgebra of k_{27} where we remove the -2 and -1 node. This is then precisely the affine algebra $\hat{\mathrm{su}}(24)$, in nice agreement with the analysis above.

4. Kac-Moody symmetries in the superstring

The construction of the previous section generalises easily to the case of the superstring, and we shall therefore be slightly sketchy here. For simplicity we will restrict our attention to finding symmetries that act on the NS-NS sector of the theory only.

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The physical state conditions in the NS-NS sector are given by

$$L_0\Psi = \bar{L}_0\Psi = \frac{1}{2}\Psi\,, \tag{4.1}$$

as well as

$$L_n\Psi = \bar{L}_n\Psi = 0 \qquad \text{for all } n > 0 \tag{4.2}$$

and

$$G_r\Psi = \bar{G}_r\Psi = 0 \qquad \text{for all } r > 0\,. \tag{4.3}$$

The physical states can be constructed in terms of the DDF operators in a similar way to the closed bosonic string. We first enlarge the space of states by doubling the zero modes, and then introduce the chiral vertex operators in terms of which the DDF-operators are constructed. Since we are working in the NS-NS sector the zero modes are the same as those of the bosonic theory and thus the construction is completely analogous. In addition to the bosonic DDF-operators A^i_n and $\bar{A}^i_n$, there are now also fermionic operators B^i_r and $\bar{B}^i_r$ where $r \in \mathbb{Z}+1/2$ and $i = 1,\ldots,8$ denotes the transverse directions. All of these operators (anti-)commute with L_m and G_r, and therefore map states satisfying equations (4.1)–(4.3) into themselves. However, as before in the bosonic case, apart from the trivial operators A^i_0 and $\bar{A}^i_0$, the single DDF operators do not leave the condition $p_L = p_R$ invariant.

The commutation relations among the different DDF operators are given by

$$\begin{aligned}
[A^i_m, A^j_n] &= m\,\delta^{ij}\,\delta_{m,-n}\,,\\
[A^i_m, B^j_r] &= 0\,,\\
\{B^i_r, B^j_s\} &= \delta^{ij}\,\delta_{r,-s}\,,
\end{aligned} \tag{4.4}$$

together with similar relations for the right-movers. Left- and right-moving DDF operators again commute or anti-commute as appropriate.

Since the commutation relations for the bosonic DDF operators are as before in the bosonic case, we can construct the generators R^{ij}_n that satisfy the commutation relations of the affine Kac-Moody algebra $\hat{su}(8) \oplus \hat{u}(1)$. The simplest fermionic DDF operators that preserve the momentum matching condition are again bilinear in the fermionic DDF-operators, and are given by

$$\begin{aligned}
F^{ij}(r) &= B^i_{-r}B^j_r - \frac{1}{2}\delta^{ij}\,,\\
\bar{F}^{ij}(r) &= \bar{B}^i_{-r}\bar{B}^j_r - \frac{1}{2}\delta^{ij}\,,\\
G^{ij}(s) &= iB^i_{-s}\bar{B}^j_{-s}\,,
\end{aligned} \tag{4.5}$$

where the index r of the first two sets of generators is assumed to be positive, while s is unrestricted. These generators are the fermionic analogues of $L^{ij}(n)$, $\bar{L}^{ij}(n)$ and $K^{ij}(n)$ of equations (3.20) and (3.21), respectively. Using the anti-commutation relations of equation (4.4) we find that they obey the commutation relations

$$\begin{aligned}
[F^{ij}(r), F^{kl}(s)] &= \delta_{r,s}\left(\delta^{jk}F^{il}(r) - \delta^{il}F^{kj}(r)\right),\\
[F^{ij}(r), G^{kl}(s)] &= \delta_{r,s}\delta^{jk}G^{il}(s) - \delta_{r,-s}\delta^{ik}G^{jl}(s)\,,
\end{aligned} \tag{4.6}$$

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as well as

$$[\bar{F}^{ij}(r), \bar{F}^{kl}(s)] = \delta_{r,s}\left(\delta^{jk}\bar{F}^{il}(r) - \delta^{il}\bar{F}^{kj}(r)\right),$$
$$[\bar{F}^{ij}(r), G^{kl}(s)] = \delta_{r,s}\delta^{jl}G^{ki}(s) - \delta_{r,-s}\delta^{il}G^{kj}(s)\,, \tag{4.7}$$

and

$$[G^{ij}(r), G^{kl}(s)] = \delta_{r,-s}\left(\delta^{ik}\bar{F}^{jl}(r) + \delta^{jl}F^{ik}(r)\right). \tag{4.8}$$

Here $r > 0$, while s is positive in the first lines of (4.6) and (4.7), and unrestricted otherwise. These commutation relations are identical to those of $L^{ij}(n)$, $\bar{L}^{ij}(n)$ and $K^{ij}(n)$. As such, these generators give rise, for each positive r, to the Lie algebra of $\mathrm{su}(16) \oplus \mathrm{u}(1)$. The generators $F^{ij}(r)$ and $\bar{F}^{ij}(r)$ define the algebra $\mathrm{u}(8) \oplus \mathrm{u}(8)$, and the generators $G^{ij}(s)$ with $s = \pm r$ enhance this algebra to $\mathrm{u}(16) = \mathrm{su}(16) \oplus \mathrm{u}(1)$.

In the type-II superstring theories, the physical states are required to be even under the two GSO-projections. While the $F^{ij}(r)$ and $\bar{F}^{ij}(r)$ generators commute with the GSO-operators $(-1)^F$ and $(-1)^{\bar{F}}$, the generator G^{ij} anti-commutes with both of them. Thus in the GSO-projected type-IIA/IIB theory only the generators $F^{ij}(r)$ and $\bar{F}^{ij}(r)$ map physical states into one another. Thus for each positive r, the relevant algebra is $\mathrm{u}(8) \oplus \mathrm{u}(8)$.

As before, we could consider the generators $F^{ij}(r)$ and $\bar{F}^{ij}(r)$ separately, but it is perhaps more natural to take the linear combination of generators that acts symmetrically on left- and right-moving states. The corresponding generators form the diagonal subalgebra, and are given as

$$S^{ij}(r) = F^{ij}(r) + \bar{F}^{ij}(r)\,. \tag{4.9}$$

As for the closed bosonic string, we may combine the different copies of the algebra $\mathrm{u}(8)$ (for different values of r) into an affine Kac-Moody algebra by considering the infinite linear combination

$$S^{ij}_m = \sum_{r=1/2<}^{\infty} \zeta^{rm}\, S^{ij}(r)\,. \tag{4.10}$$

These modes then satisfy the commutation relations of $\hat{\mathrm{su}}(8) \oplus \hat{\mathrm{u}}(1)$.

The modes R^{ij}_m and S^{ij}_m act separately on the states generated by the bosonic and fermionic DDF-operators, respectively. It is therefore natural to combine these modes as

$$U^{ij}_m = R^{ij}_m + S^{ij}_m\,. \tag{4.11}$$

The zero mode U^{ij}_0 then acts geometrically by construction, and the complete set of modes defines the affine Kac-Moody algebra $\hat{\mathrm{su}}(8) \oplus \hat{\mathrm{u}}(1)$. Following the same argument as for the closed bosonic string, we expect this algebra to be a symmetry of the perturbative string scattering amplitudes.

4.1 Comparison to perturbative symmetries

As before for the case of the bosonic string we can now compare the symmetry we have constructed above with what one finds based on the conjecture that M-theory is invariant under an e_{11} symmetry. As we have shown in section 2, the perturbative subalgebra is different for the type-IIA and -IIB superstring theories. In the former case, the perturbative sub-algebra is the very-extended su(8) algebra. Taking into account that the DDF construction only preserves the Lorentz symmetries of the transverse directions, we should then expect to find the subalgebra of the very-extended su(8) algebra where we remove the left most two dots of the Dynkin diagram in figure 2 (that are associated to the light-cone directions). The resulting algebra is then $\hat{su}(8)$, in nice agreement with the perturbative symmetry found above.

For the IIB theory, the perturbative sub-algebra implied by e_{11} is very-extended e_7. Following the same procedure as above, the part of this symmetry that is preserved by the transverse Lorentz transformations is then $\hat{e_7}$. Since e_7 contains su(8), it follows that $\hat{e_7}$ contains the affine $\hat{su}(8)$ algebra. Hence, we find that the perturbative symmetry found above in the NS-NS sector of the IIB string is a large part of the symmetry that is predicted by the e_{11} conjecture.

Actually, on general grounds one may not necessarily expect to find complete agreement between the two symmetry algebras, even once one has taken account of the non-Lorentz covariant nature of the DDF construction. First of all, the arguments of [12] only suggest that M-theory is invariant under e_{11}, but is quite possible that this is in fact part of some larger symmetry based for example on a Borcherds algebra. Secondly, the above perturbative calculation only dealt with the NS-NS sector of the theory. It is therefore possible that some of the symmetries found above may not actually lift to the full theory; conversely there may be additional symmetries that exchange states between the NS-NS and R-R sectors. Given these considerations, the extent of the agreement between the perturbative symmetries following from the e_{11} conjecture, and those found in terms of the DDF construction is substantial and gives support for the proposal that e_{11} is indeed a symmetry of M-theory.

5. Conclusions and discussions

In this paper we have tested some predictions of the conjecture that M-theory and the bosonic string possesses e_{11} and k_{27} as symmetries, respectively. More specifically, we have determined the perturbative subalgebras for type IIA, IIB and the closed bosonic string that are predicted by this proposal, and we have compared them with the symmetry algebras that can be constructed in perturbative string theory, using the DDF construction. Taking account of the non-Lorentz covariance of the DDF-construction, we have found remarkable agreement.

At first sight it seems odd that the perturbative string theories should have an (infinite dimensional) affine symmetry algebra, given that there are only finitely many states at each mass level. However, the affine algebras that appear in our construction are at level zero (i.e. have a trivial centre), and such algebras do possess non-trivial finite-dimensional representations.

The symmetry algebras that have been proposed for M-theory and the closed bosonic string, e_{11} and k_{27}, are rather special lorentzian algebras. In fact both are very-extended lorentzian Kac-Moody algebras whose finite dimensional semi-simple Lie algebras, e_8 and

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d_{24}, are closely related to even self-dual euclidean lattices [17]. As is well known even self-dual euclidean lattices only exist in dimensions $D = 8n$, where $n = 1, 2, 3, \ldots$. The first example therefore arises in dimension eight, where there is a unique possibility, the root lattice of e_8. The corresponding very-extended algebra is then e_{11}. In dimension $D = 24$, on the other hand, there are twenty-four such lattices, the so-called Niemeier lattices [18]. Apart from the Leech lattice that does not have any vectors of length squared two, all other Niemeier lattices are associated with finite dimensional semi-simple Lie algebras. The root lattices of these Lie algebras are in general not self-dual; however, they can be made self-dual by addition of a specific set of conjugacy classes of the corresponding weight lattice. Only two of the Lie algebras that arise in this context are simple, namely d_{24} and a_{24}. As we have seen above, the very-extended algebra corresponding to the former Lie algebra is k_{27} that has been proposed as the symmetry algebra of the bosonic string.[5] Thus, if we restrict our attention to very-extended algebras that are associated to even self-dual lattices in the way described above, we find almost uniquely the conjectured symmetry algebras of M-theory and the twenty-six dimensional bosonic string. This suggests that even self-dual lattices play an important rôle in the proper formulation of M-theory and the analogous theory in twenty-six dimensions.

It is interesting to speculate about the significance of some of the other symmetry algebras that can be constructed in a similar vein. If instead of d_{24} we considered a_{24}, the corresponding very-extended Kac-Moody algebra would contain $a_{26} = \mathrm{su}(27)$ as a subalgebra. If this very-extended Kac-Moody algebra was the symmetry algebra of some theory, this theory would be expected to contain gravity in twenty-seven dimensions. In fact, the corresponding theory would have a low energy effective action that is probably just twenty-seven dimensional gravity, since gravity in D dimensions is believed to have the very-extended a_D Kac-Moody algebra as a symmetry [24].

5.1 An 18-dimensional theory

Given the close relationship between the root lattices of d_{8n} for $n = 1, 3$ and the conjectured symmetry algebras of M-theory and the closed bosonic string, respectively, it is also interesting to consider the case of the symmetry algebra that is associated to the root lattice of d_{16}. As before, the corresponding even self-dual lattice is obtained by adding to the root lattice of d_{16} the conjugacy class of weight vectors that contains one of the two spinor weights. (By the way, apart from the root lattice of $e_8 \oplus e_8$ this is the only even self-dual lattice in 16 dimensions.) Let us denote the corresponding very-extended Kac-Moody algebra by k_{19}. Its Dynkin diagram is of the same form as that of figure 4, except that there are only 17 nodes along the horizontal.

Given the k_{19} symmetry we can deduce the low energy effective action of the proposed new theory, using similar arguments to those in [12]. The nodes along the horizontal line are just those of a_{17} and therefore correspond to the presence of a graviton. The vertical

[5] It might seem that the symmetries of the closed bosonic string and M-theory arise in a different way. However, the root lattice of e_8 can be obtained from the root lattice of d_8 by addition of one of the spinor conjugacy classes of d_8. From this point of view, the construction of the root lattice of e_8 and the Niemeier lattice corresponding to d_{24} is completely analogous.

node on the right corresponds to a rank two anti-symmetric tensor of a_{17} which we identify with a gauge field, $B_{a_1 a_2}$, while the vertical node to the left corresponds to a rank fourteen anti-symmetric tensor that we denote by $B_{a_1 \ldots a_{14}}$; it can be identified with the dual gauge field of the anti-symmetric tensor $B_{a_1 a_2}$.

By construction the algebra k_{19} has rank nineteen. The graviton is associated to the group GL(18), and its associated generators provide the Cartan subalgebra of a_{17}, as well as one additional Cartan generator D of k_{19}. The remaining Cartan generator, which we denote by R, corresponds then to an additional scalar field, denoted by ϕ. The low energy effective action of the proposed new theory in eighteen dimensions therefore contains gravity, a second rank tensor gauge field $B_{a_1 a_2}$, its dual gauge field $B_{a_1 \ldots a_{14}}$, a scalar ϕ and its dual $\phi_{a_1 \ldots a_{16}}$. The field content is therefore very analogous to that of the closed bosonic string in twenty-six dimensions.

A theory with just such a field content was constructed as a non-linear realisation in reference [12]. The relevant Lie algebra is generated by $K^a{}_b$, R, $R^{a_1 a_2}$, $R^{a_1 \ldots a_{14}}$ and $R^{a_1 \ldots a_{16}}$, with commutation relations

$$\begin{aligned} [K^a{}_b, R^{a_1 \ldots a_p}] &= \delta^{a_1}{}_b R^{b a_2 \ldots a_p} + \cdots , \\ [R, R^{a_1 \ldots a_p}] &= c_p \, R^{a_1 \ldots a_p} , \\ [R^{a_1 a_2}, R^{a_1 \ldots a_{14}}] &= 2 \, R^{a_1 \ldots a_{16}} , \end{aligned} \tag{5.1}$$

where $c_2 = 1/4 = -c_{14}$, $c_0 = 0 = c_{16}$, and all other commutators vanish. The corresponding effective action, after the elimination of the dual fields from the equations of motion, is then given by

$$S = \int d^{18}x \left(R - \frac{1}{2} \, \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{12} \, e^{\beta\phi} \, H_{\mu\nu\lambda} \, H^{\mu\nu\lambda} \right) , \tag{5.2}$$

where $\beta = 1/\sqrt{2}$.

It is encouraging that upon dimensional reduction of this effective action on a torus some part of the underlying Kac-Moody symmetry becomes manifest. Indeed, it was shown in [24] that this is the case provided that β is chosen as above. Although the class of actions that give rise to Kac-Moody symmetries upon reduction is very restricted, the generalisation of the above action to D dimensions also leads to a Kac-Moody symmetry provided one chooses $\beta = \sqrt{8/(D-2)}$. It is an amusing coincidence that for $D = 8n + 2$ this expression simplifies to $\beta = \sqrt{1/n}$. Furthermore, for $n = 1, 2, 3$ the above action describes the (gravity, scalar and anti-symmetric 2-form sector) of the low energy effective action for the ten dimensional type-II superstring, the eighteen dimensional string theory proposed here, and the twenty-six dimensional closed bosonic string theory, respectively.

Next we want to show that the Kac-Moody algebra that is associated with this theory is in fact k_{19}. Our arguments will be similar to those that were used to deduce that e_{11} and k_{27} are the symmetries for the maximal supergravity theory in eleven dimension and the closed bosonic string theory in twenty-six dimensions, respectively. If the theory is to admit a Kac-Moody algebra as a symmetry then the generators

$$K^a{}_b \, , \quad a < b \, , \qquad R^{a_1 a_2} \, , \qquad R^{a_1 \ldots a_{14}} \, , \qquad R^{a_1 \ldots a_{16}} \tag{5.3}$$

must be some of its positive roots. Furthermore, we can expect its Cartan sub-algebra to contain

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}\,, \quad a = 1, \dots, 17\,, \qquad D = \sum_{a=1}^{18} K^a{}_a\,, \quad R\,. \tag{5.4}$$

The set of equation (5.3) is generated by the multiple commutators of

$$E_a = K^a{}_{a+1}\,, \quad a = 1, \dots, 17\,, \qquad E_{18} = R^{1718}\,, \qquad E_{19} = R^{5\dots18}\,, \tag{5.5}$$

which, as the notation suggests, we should identify with the simple roots of the proposed Kac-Moody algebra. Next we want to find Cartan generators (that can be expressed in terms of the generators of equation (5.4)), which lead to an acceptable Cartan matrix for a Kac-Moody algebra and contain the subalgebra a_{17} in the appropriate way. One can readily show that the choice

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1}\,, \qquad a = 1, \dots, 17\,, \\ H_{18} &= K^{17}{}_{17} + K^{18}{}_{18} - \frac{1}{8}D + R\,, \\ H_{19} &= K^5{}_5 + \dots + K^{18}{}_{18} - \frac{7}{8}D - R \end{aligned} \tag{5.6}$$

satisfies these requirements. It follows from these equations and the Kac-Moody algebra relation $[H_a, E_b] = A_{ab}E_b$ that the Cartan matrix A_{ab} is that for the very-extended d_{16} Kac-Moody algebra, i.e. k_{19}.

Thus we have found convincing evidence that, should there exists a theory in eighteen dimensions that is associated with the very-extended d_{16} algebra, then this theory would have a low energy effective action that is given by equation (5.2). A graviton and second rank tensor gauge field in D dimensions have $\frac{1}{2}(D-1)(D-2) - 1$ and $\frac{1}{2}(D-2)(D-3)$ degrees of freedom, respectively. Together with the single scalar field we therefore find that the number of bosonic degrees of freedom is 256. If the theory were supersymmetric we would require a matching number of fermionic degrees of freedom. One of these fermions would have to be the gravitino which has $2^{\frac{(D-2)}{2}}(D-3)r$ degrees of freedom, where $r = 1/2$ or $r = 1$ if it is a Majorana-Weyl or Majorana spinor, respectively. For $D = 18$ we find that the gravitino has $\frac{15}{2} \cdot 256$ degrees of freedom even if it were a Majorana-Weyl spinor, and we may conclude that this theory could not possess space-time supersymmetry.

Given the presence of the rank two tensor gauge field and the graviton it would seem reasonable to suppose that this theory was a type of string theory. Indeed the count of states is just that for a closed bosonic string theory. Clearly, this theory would posses an anomaly and one would have to add a conformal field theory which did not contribute states in the massless sector and lead to a modular invariant partition function. Some time ago an eighteen dimensional string theory has been suggested [25] within the context of the mechanism proposed for recovering supersymmetric string theories from the closed bosonic string in twenty-six dimensions. However, the theory of [25] has only 136 massless bosons, and therefore does not seem to be the same as that considered here.

5.2 Relation to Borcherds algebras of even self-dual lorentzian lattices

Finally, let us comment on the relation of the symmetry algebras we have found to Borcherds algebras [10]. Recall that to every even self-dual euclidean lattice Λ of dimension D, one can naturally associate an even self-dual lorentzian lattice of dimension $D+2$. This lattice is simply $\Lambda \oplus \mathrm{II}^{1,1}$, where $\mathrm{II}^{1,1}$ denotes the even self-dual lorentzian lattice in $1+1$ dimensions. Even self-dual lorentzian lattices exist in dimensions $D = 8n+2$, and for each n there is a unique such lattice that is usually denoted by $\mathrm{II}^{1,8n+1}$. It is easy to see that if Λ is the root lattice of a semi-simple Lie algebra, then $\Lambda \oplus \mathrm{II}^{1,1}$ is the root lattice of the over-extension of this algebra [26]. As we have seen above, the symmetry algebras e_{11} and k_{27} originate from finite dimensional semi-simple Lie algebras that are associated with even self-dual euclidean lattices. The corresponding over-extensions are therefore related to even self-dual lorentzian lattices, and the very-extended algebras that are of relevance here can be obtained from these by adding one additional node to the Dynkin diagram.

For the case of the symmetries of M-theory, the relevant over-extended algebra is the hyperbolic algebra e_{10}, whose root lattice is the unique even self-dual lorentzian lattice $\mathrm{II}^{1,9}$. It has been known for some time that the hyperbolic Lie algebra e_{10} is actually rather complicated — for example its root multiplicities are only known at low levels. On the other hand, the corresponding Borcherds algebra, i.e. the algebra of physical states of the bosonic string compactified on the lattice $\mathrm{II}^{1,9}$, has known root multiplicities and contains e_{10} [27, 28]. One may therefore guess that the actual symmetry algebra of M-theory may indeed contain this Borcherds algebra rather than just e_{10}.

Similarly, for the case of the closed bosonic string, the over-extension of d_{24} is a 26-dimensional lorentzian Kac-Moody algebra k_{26}. Its root lattice is the sublattice $\Lambda_{d_{24}} \oplus \mathrm{II}^{1,1}$ of the unique even self-dual lorentzian lattice $\mathrm{II}^{1,25}$. From the point of view of twenty-four dimensions, the corresponding Niemeier lattice $\Lambda^N_{d_{24}}$ can be obtained from the root lattice of d_{24} by adding the conjugacy class of the d_{24} weight space that contains one of the spinors. Since the shortest vector in this conjugacy class has length squared six, it is conceivable that these vectors are not visible in the low-energy analysis that underlies [12], and that the symmetry algebra should actually contain all the roots of $\mathrm{II}^{1,25}$. Furthermore, this would then suggest that the actual symmetry algebra contains the corresponding Borcherds algebra, which, in this case, is just the fake Monster algebra. The fake monster algebra was suggested as a symmetry of the closed bosonic string compactified on the torus $\mathrm{II}^{1,25}$ [29, 14]. The philosophy of [12] is that symmetries that are seen upon dimensional reduction are actually symmetries of the underlying unreduced theory. It is an encouraging sign that the symmetry algebra k_{27} that was found from the low energy effective action in twenty-six dimensions, contains the algebra k_{26} which, as we have just explained, is closely related to the fake Monster symmetry that emerges upon dimensional reduction.

In the above we have suggested that we should replace the over-extended subalgebra of the symmetry algebra by the Borcherds algebra associated to the corresponding even self-dual (lorentzian) lattice. This then also implies that the full symmetry algebra is not just a very-extended Kac-Moody algebra, but actually some extension of a Borcherds algebra. For the case of the superstring it is conceivable that the relevant algebra is just the usual

Borcherds algebra based on the root lattice of e_{11}, but this is not possible for k_{27} since the dimension of the latter lattice is 27 and thus the no-ghost theorem of string theory (that plays a crucial role for Borcherds' construction) fails. At any rate, it seems unlikely that the full symmetry algebra is just a Borcherds algebra, given that the underlying theories in 11 and 27 dimensions are probably theories of membranes rather than strings. Thus we expect that the very-extended symmetry algebras should be replaced by some generalisation of Borcherds algebras whose vertex operators are based on membranes. It would be very interesting to discover what class of algebras membrane vertex operators lead to.

The idea that the underlying algebras are some sort of Borcherds algebras is also attractive from a more conceptual point of view. As is well known, the root lattices of inequivalent Kac-Moody algebras can be the same (for some explicit examples see for example [17]), and thus the root lattice does not uniquely define the corresponding Kac-Moody algebra. Rather, in order to define the Kac-Moody algebra, one must also specify a preferred set of basis vectors, namely the simple roots whose scalar products define the Cartan matrix of the Kac-Moody algebra. On the other hand, the Borcherds algebras can be uniquely constructed from their root lattices (they are just the algebra of physical states of the string compactification on the corresponding lattice), and their root multiplicities are known (since they are just the number of physical states of a given momentum). It would be very interesting to find more evidence for these speculations.

A. The simple roots of e_8

The root lattice of e_8 is an eight-dimensional euclidean even self-dual lattice. It is spanned by the simple roots which can be taken to be given by the following vectors in $\mathbb{R}^8$.

$$\begin{aligned}
\alpha_1 &= (0,0,0,0,0,1,-1,0)\,,\\
\alpha_2 &= (0,0,0,0,1,-1,0,0)\,,\\
\alpha_3 &= (0,0,0,1,-1,0,0,0)\,,\\
\alpha_4 &= (0,0,1,-1,0,0,0,0)\,,\\
\alpha_5 &= (0,1,-1,0,0,0,0,0)\,,\\
\alpha_6 &= (-1,-1,0,0,0,0,0,0)\,,\\
\alpha_7 &= \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\\
\alpha_8 &= (1,-1,0,0,0,0,0,0)\,.
\end{aligned} \tag{A.1}$$

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